

# POSITIVE BIORTHOGONAL CURVATURE ON $S^2 \times S^2$

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**ABSTRACT.** We construct metrics on  $S^2 \times S^2$ , arbitrarily close to the standard product metric, for which the average of the sectional curvatures of any two orthogonal 2-planes is strictly positive. Although these metrics have positive Ricci curvature, they do not have nonnegative sectional curvature.

## 1. INTRODUCTION

Let  $(M, g)$  be a 4-dimensional Riemannian manifold. For each plane  $\sigma \subset T_p M$  at a point  $p \in M$ , denote by  $\sigma^\perp$  the orthogonal plane to  $\sigma$ , i.e.,  $\sigma \oplus \sigma^\perp = T_p M$  is a  $g$ -orthogonal direct sum. Define the *biorthogonal (sectional) curvature* of  $\sigma$  as the average of the sectional curvatures of  $\sigma$  and  $\sigma^\perp$ , i.e.,

$$\sec_g^\perp(\sigma) := \frac{1}{2}(\sec_g(\sigma) + \sec_g(\sigma^\perp)).$$

The Hopf Conjecture, that asks if  $S^2 \times S^2$  admits a metric with  $\sec > 0$ , remains one of the most intriguing open problems in Riemannian geometry. With the standard product metric  $g_0$ , at every point  $p \in S^2 \times S^2$  there exists  $\sigma \subset T_p M$  with  $\sec_{g_0}^\perp(\sigma) = 0$ . Namely, any *mixed plane*  $\sigma$  at  $p$  (i.e., spanned by vectors of the form  $(X, 0)$  and  $(0, Y)$ ) is such that  $\sigma^\perp$  is also a mixed plane, hence  $\sec_{g_0}(\sigma) = \sec_{g_0}(\sigma^\perp) = 0$ . A natural question in this context is if the weaker condition  $\sec^\perp > 0$  can be satisfied in  $S^2 \times S^2$  [2]. The goal of this note is to give a positive answer:

**Theorem.** *There exist smooth Riemannian metrics  $g_*$  on  $S^2 \times S^2$  with  $\sec_{g_*}^\perp > 0$ , arbitrarily close to the standard product metric  $g_0$  in the  $C^k$ -topology,  $k \geq 1$ .*

Our metrics  $g_*$  are obtained as arbitrarily  $C^k$ -small deformations of  $g_0$ . In particular, it follows that there exists a sequence  $\{g_n\}$  of metrics with  $\sec^\perp > 0$  on  $S^2 \times S^2$  such that  $\{g_n\}$  converges to  $g_0$  in the  $C^k$ -topology,  $k \geq 1$ . As a consequence, our metrics with positive biorthogonal curvature also have positive Ricci curvature, provided they are chosen sufficiently close to  $g_0$ . Notice this is not a general fact, e.g., the standard product metric on  $S^1 \times S^3$  has  $\sec^\perp > 0$ , however this manifold does not support any metric with positive Ricci curvature. In general, positive biorthogonal curvature only implies positive scalar curvature. Moreover, we remark that  $(S^2 \times S^2, g_*)$  has planes at many points with zero curvature (and even some with negative curvature), however any two of them are never orthogonal. It would be interesting to know if metrics with  $\sec^\perp > 0$  on  $S^2 \times S^2$  can also be constructed while keeping  $\sec \geq 0$ .

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Metric deformations to improve curvature have a long history, stemming from Berger and his students in the 1970's to the recent construction proposed by Petersen and Wilhelm [5, 6] of a positively curved exotic sphere. Of particular importance in the present note are techniques developed by Müter [4] and Strake [9, 10], respectively regarding Cheeger deformations and deformations positive of first-order. The *Cheeger deformation* is a method to attempt to increase curvature on nonnegatively curved manifolds with symmetries, by shrinking the metric in the direction of orbits of a large isometry group. This technique was introduced by Cheeger [1], inspired by the construction of Berger metrics on odd-dimensional spheres, where the round metric is shrunk in the direction of the Hopf fibers. Müter [4] carried out a systematic study of Cheeger deformations in his PhD thesis under W. Meyer, establishing ground for a much better understanding of these deformed metrics. Strake [10], another PhD student of W. Meyer during the same period, studied metric deformations of nonnegatively curved metrics for which the first variation of the sectional curvature of any zero curvature plane is positive. These deformations are called *positive of first-order*, and if the manifold is compact, they yield actual positively curved metrics. They also observed that, in this infinitesimal sense, Cheeger deformations are *nonnegative* of first-order.

Our deformation process from the product metric  $g_0$  to a metric  $g_*$  of positive biorthogonal curvature has two steps, in which the above techniques are combined. The first is a Cheeger deformation, described in detail by Müter [4, 11]. More precisely, we consider the cohomogeneity one diagonal  $\mathrm{SO}(3)$ -action on  $S^2 \times S^2$  and shrink  $g_0$  in the direction of the orbits. This deformation gives a family of metrics  $g_t$ ,  $t > 0$ , with  $\sec_{g_t} \geq 0$  and much fewer planes of zero curvature than  $g_0$ . Namely,  $(S^2 \times S^2, g_t)$  has a circle's worth of zero curvature planes on points that lie on the diagonal or the anti-diagonal  $\pm\Delta S^2 = \{(p, \pm p) : p \in S^2\} \subset S^2 \times S^2$ , and a unique zero curvature plane at any other point. This means that  $\sec^\perp \geq 0$ , and equality holds only for some planes whose base point is in one of the submanifolds  $\pm\Delta S^2$ .

Next, for fixed  $t > 0$ , let  $g := g_t$ . The second step is to employ a first-order local conformal deformation  $g_s = g + s h$ , where  $h = \phi g$ , and  $\phi$  is supported in a tubular neighborhood of  $\pm\Delta S^2$ . In our setup, given the geometry of  $(S^2 \times S^2, g)$ , we construct  $\phi$  such that  $\frac{d}{ds} \sec_{g_s}^\perp(\sigma)|_{s=0} > 0$  for all planes  $\sigma$  with  $\sec_g^\perp(\sigma) = 0$ . The function  $\phi$  is proportional to the square distance to  $\pm\Delta S^2$  in the metric  $g$ , multiplied by a cutoff function. The strategy for such a construction is adapted from Strake [9, 10]. Finally, a standard compactness argument implies that  $\sec_{g_s}^\perp > 0$  for all sufficiently small  $s > 0$ , proving the desired result.

This paper is organized as follows. In Section 2, we review basic aspects of Cheeger deformations, following Müter [4, 11]. We describe the metric on  $S^2 \times S^2$  obtained by a Cheeger deformation with respect to the diagonal  $\mathrm{SO}(3)$ -action in terms of its biorthogonal curvature. In Section 3, we analyze the effects of a first-order deformation and construct an appropriate variation to give positive biorthogonal curvature on  $S^2 \times S^2$ , starting from the Cheeger deformed metric. Finally, a few final remarks on the geometry of the constructed metrics are given in Section 4, including a proof that our metrics  $g_*$  do not have  $\sec_{g_*} \geq 0$ .

## 2. FIRST STEP: CHEEGER DEFORMATION

Although the techniques used in this section are mostly available elsewhere in the literature, see [4, 11, 12], we briefly recall a few basic aspects as a service to the reader. For convenience, we use the same notation as the above references.

**2.1. Cheeger deformation.** Let  $(M, g)$  be a Riemannian manifold and  $G$  a compact Lie group that acts on  $M$  by isometries. The *Cheeger deformation* of  $g$  is a 1-parameter family  $g_t$ ,  $t \geq 0$ , of  $G$ -invariant metrics on  $M$ , defined as follows. Let  $Q$  be a bi-invariant metric on  $G$ , and endow  $M \times G$  with the product metric  $g + \frac{1}{t}Q$ . Consider the submersion

$$(2.1) \quad \rho: M \times G \rightarrow M, \quad \rho(p, h) = h^{-1}p,$$

and define  $g_t$  as the metric on  $M$  that turns  $\rho$  into a Riemannian submersion. The family of metrics  $g_t$  extends smoothly across  $t = 0$ , with  $g_0 = g$ , thus providing a deformation of such metric. Since  $\sec_Q \geq 0$ , it follows immediately from the O'Neill formula that, if  $\sec_{g_0} \geq 0$ , then also  $\sec_{g_t} \geq 0$ ,  $t \geq 0$ . As we will see, many planes with zero curvature with  $g_0$  usually gain positive curvature with  $g_t$ .

For each  $p \in M$ , denote by  $G_p$  the isotropy group at  $p$  and by  $\mathfrak{g}_p$  its Lie algebra. Fix the  $Q$ -orthogonal splitting  $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{m}_p$ , and identify  $\mathfrak{m}_p$  with the tangent space  $T_p G(p)$  to the  $G$ -orbit through  $p$  via action fields. More precisely, we identify  $X \in \mathfrak{m}_p$  with  $X_p^* = \frac{d}{ds} \exp(sX)p|_{s=0} \in T_p G(p)$ . This determines a  $g_t$ -orthogonal splitting  $T_p M = \mathcal{V}_p \oplus \mathcal{H}_p$  in *vertical space*  $\mathcal{V}_p := T_p G(p) = \{X_p^* : X \in \mathfrak{m}_p\}$  and *horizontal space*  $\mathcal{H}_p := \{v \in T_p M : g_t(v, \mathcal{V}_p) = 0\}$ . Notice that the dimensions of  $\mathcal{V}_p$  and  $\mathcal{H}_p$  may vary with  $p \in M$ , hence these are not distributions.

Let  $P_t: \mathfrak{m}_p \rightarrow \mathfrak{m}_p$  be the  $Q$ -symmetric automorphism that relates the metrics  $Q$  and  $g_t$ , i.e., such that

$$(2.2) \quad Q(P_t(X), Y) = g_t(X_p^*, Y_p^*), \quad X, Y \in \mathfrak{m}_p.$$

It is an easy computation that  $P_t$  is determined by  $P_0$  in the following way:

$$(2.3) \quad P_t = (P_0^{-1} + t \text{Id})^{-1} = P_0 (\text{Id} + tP)^{-1}, \quad t \geq 0,$$

see [11, Prop 1.1]. Thus, if we let  $C_t: T_p M \rightarrow T_p M$  be the  $g$ -symmetric automorphism that relates  $g$  and  $g_t$ , i.e., such that

$$(2.4) \quad g(C_t(X), Y) = g_t(X, Y), \quad X, Y \in T_p M,$$

we then get

$$(2.5) \quad C_t(X) = P_0^{-1} P_t(X^\mathcal{V}) + X^\mathcal{H}, \quad X \in T_p M,$$

where  $X^\mathcal{V}$  and  $X^\mathcal{H}$  are the vertical and horizontal components of  $X$  respectively. This reveals how the geometry of  $g_t$  changes with  $t$ , since if  $P_0$  has eigenvalues  $\lambda_i$ , then  $C_t$  has eigenvalues  $\frac{1}{1+t\lambda_i}$  corresponding to the vertical directions and eigenvalues 1 in the horizontal directions. In other words, as  $t$  grows, the metric  $g_t$  shrinks in the direction of the orbits and remains the same in the orthogonal directions.

**2.2. Curvature evolution.** Let us now analyze how the curvature changes under this deformation. Henceforth, we assume the initial metric  $g_0$  has  $\sec_{g_0} \geq 0$ . As explained above, this implies  $\sec_{g_t} \geq 0$  for all  $t \geq 0$ . Given  $X \in T_p M$ , denote by  $X_\mathfrak{m}$  the unique vector in  $\mathfrak{m}_p$  such that  $(X_\mathfrak{m})_p^* = X_p^\mathcal{V}$ . Also, given a plane  $\sigma = \text{span}\{X, Y\}$ , we write

$$C_t^{-1}(\sigma) := \text{span}\{C_t^{-1}X, C_t^{-1}Y\}.$$

As explained by Ziller [11], the crucial observation of Mütter is that, to analyze the evolution of  $\sec_{g_t}$ , it is much more convenient to study  $\sec_{g_t}(C_t^{-1}(\sigma))$  rather than  $\sec_{g_t}(\sigma)$ . In more recent literature, the 1-parameter family of bundle automorphisms induced by  $C_t^{-1}$  in the Grassmannian bundle  $Gr_2TM$  of 2-planes on  $M$  is being called *Cheeger reparametrization*, see [5, 6]. The following result of Mütter [4, Satz 3.10] (see also [11, Cor 1.4]) summarizes how the curvature of  $g_t$  evolves.

**Proposition 2.1.** *Let  $g_t$  be the Cheeger deformation of  $g_0$ ,  $\sec_{g_0} \geq 0$ . Given  $\sigma = \text{span}\{X, Y\} \subset T_p M$ , consider the unnormalized  $g_t$ -sectional curvature of  $C_t^{-1}(\sigma)$ :*

$$k_c(t) := \|C_t^{-1}X \wedge C_t^{-1}Y\|_{g_t}^2 \sec_{g_t}(C_t^{-1}(\sigma)) = g_t(R_t(C_t^{-1}X, C_t^{-1}Y)C_t^{-1}Y, C_t^{-1}X),$$

where  $R_t$  is the curvature tensor of  $g_t$ . If  $\sec_{g_0}(\sigma) = 0$ , then:

- (i)  $k'_c(0) \geq 0$ ;
- (ii) If  $k'_c(0) = 0$  and  $[X_m, Y_m] \neq 0$ , then  $k''_c(0) = 0$ ,  $k'''_c(0) > 0$  and  $k_c(t) > 0$  for all  $t > 0$ ;
- (iii) If  $k'_c(0) = 0$  and  $[X_m, Y_m] = 0$ , then  $k_c(t) = 0$  for all  $t > 0$ .

In particular, if  $\sec_{g_0}(\sigma) = 0$  and  $[X_m, Y_m] \neq 0$  (i.e., the plane  $\text{span}\{P_0X_m, P_0Y_m\}$  has positive curvature in  $(G, Q)$ ), then  $\sec_{g_t}(C_t^{-1}(\sigma)) > 0$  for all  $t > 0$ .

Observe that, from (iii), if  $\sigma$  is tangent to a totally geodesic flat torus in  $M$  that contains a horizontal direction, then  $\sec_{g_t}(C_t^{-1}(\sigma)) = 0$ ,  $t \geq 0$ , i.e.,  $\sigma$  remains flat. On the other hand, we also get the following important positive result:

**Corollary 2.2.** *Assume  $G = \text{SO}(3)$  or  $G = \text{SU}(2)$ , so that  $\sec_Q > 0$ . If  $\sec_{g_0}(\sigma) = 0$  and the image of the projection of  $\sigma \subset \mathcal{V}_p \oplus \mathcal{H}_p$  onto  $\mathcal{V}_p$  is 2-dimensional, then  $\sec_{g_t}(C_t^{-1}(\sigma)) > 0$  for all  $t > 0$ . In other words, up to the Cheeger reparametrization, zero curvature planes with nondegenerate vertical projection have positive curvature with  $g_t$ , for all  $t > 0$ .*

**2.3. The case of  $S^2 \times S^2$ .** Consider  $S^2 \times S^2$  endowed with the standard product metric  $g_0$  and the diagonal  $\text{SO}(3)$ -action:

$$A \cdot (p_1, p_2) = (Ap_1, Ap_2), \quad p = (p_1, p_2) \in S^2 \times S^2 \subset \mathbb{R}^3 \oplus \mathbb{R}^3, A \in \text{SO}(3).$$

This is a cohomogeneity one isometric action with orbit space a closed interval, so there are codimension one principal orbits (corresponding to interior points of the interval) and two singular orbits (corresponding to the endpoints), see [12]. These singular orbits are the *diagonal* and *anti-diagonal* submanifolds:

$$\pm\Delta S^2 := \{(p, \pm p) : p \in S^2\} \subset S^2 \times S^2.$$

The principal isotropy  $G_p$ ,  $p \notin \pm\Delta S^2$ , is trivial, since it consists of orientation-preserving isometries of  $\mathbb{R}^3$  that fix two linearly independent directions. The singular isotropies are formed by orientation-preserving isometries of  $\mathbb{R}^3$  that fix one direction, hence are isomorphic to  $\text{SO}(2)$ . Thus, the group diagram of this action is  $\{1\} \subset \{\text{SO}(2), \text{SO}(2)\} \subset \text{SO}(3)$ .

Following Mütter [4], we identify the Lie algebra of  $\text{SO}(3)$  with  $\mathbb{R}^3$  by:

$$\mathfrak{so}(3) \ni Z = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = z \in \mathbb{R}^3.$$

Considering  $(\mathfrak{so}(3), Q)$  endowed with the standard bi-invariant metric, the above is an isometric identification with Euclidean space  $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ . In this way, since the

Lie exponential in  $\mathrm{SO}(3)$  is given by matrix exponentiation, the action field induced by  $Z \in \mathfrak{so}(3)$  is:

$$(2.6) \quad Z_p^* = (Z_{p_1}^*, Z_{p_2}^*) = (Z p_1, Z p_2) = (z \wedge p_1, z \wedge p_2) \in T_p(S^2 \times S^2).$$

So, if  $x, y \in \mathbb{R}^3$  are such that  $\langle x, p_1 \rangle = \langle y, p_2 \rangle = 0$ , then for all  $z \in \mathbb{R}^3$ ,

$$(2.7) \quad g_0((X_{p_1}^*, Y_{p_2}^*), Z_p^*) = \langle x \wedge p_1, z \wedge p_1 \rangle + \langle y \wedge p_2, z \wedge p_2 \rangle = \langle x + y, z \rangle.$$

Thus, the vector  $(X_{p_1}^*, -X_{p_2}^*) \in T_p(S^2 \times S^2)$  is horizontal, whenever  $x \in \{p_1, p_2\}^\perp := \{x \in \mathbb{R}^3 : \langle x, p_1 \rangle = \langle x, p_2 \rangle = 0\}$ . By dimensional reasons, it then follows that the horizontal space for the  $\mathrm{SO}(3)$ -action on  $S^2 \times S^2$  at  $p = (p_1, p_2)$  is

$$\mathcal{H}_p = \{(X_{p_1}^*, -X_{p_2}^*) \in T_p(S^2 \times S^2) : x \in \{p_1, p_2\}^\perp\}.$$

Recall that the vertical space is  $\mathcal{V}_p = \{(X_{p_1}^*, X_{p_2}^*) : X \in \mathfrak{m}_p\}$ , see (2.6). For general  $x, y \in \mathbb{R}^3$ , analogously to (2.7), we have:

$$\begin{aligned} g_0(X_p^*, Y_p^*) &= g_0((X_{p_1}^*, X_{p_2}^*), (Y_{p_1}^*, Y_{p_2}^*)) = \langle x \wedge p_1, y \wedge p_1 \rangle + \langle x \wedge p_2, y \wedge p_2 \rangle = \\ &= \langle p_1 \wedge (x \wedge p_1), y \rangle + \langle p_2 \wedge (x \wedge p_2), y \rangle = \langle (2x - \langle x, p_1 \rangle p_1 - \langle x, p_2 \rangle p_2), y \rangle. \end{aligned}$$

From (2.2), the above is equal to  $\langle P_0 X, Y \rangle$ , so we get an explicit formula for  $P_0 : \mathfrak{m}_p \rightarrow \mathfrak{m}_p$  in our example:

$$P_0 X = 2X - \langle X, p_1 \rangle p_1 - \langle X, p_2 \rangle p_2.$$

In particular, it follows that the subspace  $\{p_1, p_2\}^\perp \subset \mathfrak{m}_p$  is invariant under  $P_0$  and hence under  $P_t$  and  $C_t$ , see (2.3) and (2.5).

Let  $\pi : \{p_1, p_2\}^\perp \rightarrow \{p_1, p_2\}^\perp / \sim$  be the projection onto the corresponding real projective space. For each  $x \in \{p_1, p_2\}^\perp$ , consider the vertizontal<sup>1</sup> plane

$$(2.8) \quad \sigma_{\pi(x)} := \mathrm{span}\{(X_{p_1}^*, 0), (0, X_{p_2}^*)\} = \mathrm{span}\{(X_{p_1}^*, X_{p_2}^*), (X_{p_1}^*, -X_{p_2}^*)\}.$$

This is the unique mixed plane at  $p$  that contains the horizontal vector  $(X_{p_1}^*, -X_{p_2}^*)$ . Thus, from Corollary 2.2,  $\{\sigma_{\pi(x)} \subset T_p(S^2 \times S^2) : x \in \{p_1, p_2\}^\perp\}$  are the only  $g_0$ -flat planes at  $p$  that remain  $g_t$ -flat for  $t > 0$ , up to the Cheeger reparametrization. As a matter of fact, by the above, the Cheeger reparametrization fixes such planes, i.e.,

$$(2.9) \quad C_t^{-1}(\sigma_{\pi(x)}) = \sigma_{\pi(x)}.$$

In conclusion, for any  $t > 0$ ,  $\sec_{g_t} \geq 0$  and  $\sec_{g_t}(\sigma) = 0$  if and only if  $\sigma = \sigma_{\pi(x)} \subset T_p(S^2 \times S^2)$  for some  $x \in \{p_1, p_2\}^\perp$ . In particular,  $g_t$ -flat planes at  $p$  are parametrized by  $\pi(\{p_1, p_2\}^\perp)$ . Thus, in  $(S^2 \times S^2, g_t)$ ,  $t > 0$ , there is a circle's worth of zero curvature planes at each  $p \in \pm \Delta S^2$ , a unique zero curvature plane at each  $p \notin \pm \Delta S^2$ , and all other planes have positive curvature.

**Proposition 2.3.** *Let  $g_t$  be the Cheeger deformation of  $(S^2 \times S^2, g_0)$  with respect to the diagonal  $\mathrm{SO}(3)$ -action. Then, for any  $t > 0$ ,  $\sec_{g_t}^\perp \geq 0$  and equality only holds for planes at  $\pm \Delta S^2$ . If  $x, x^\perp \in \{p_1\}^\perp$ ,  $\langle x, x^\perp \rangle = 0$ , then  $\sigma_{\pi(x)}$  and  $\sigma_{\pi(x^\perp)} = (\sigma_{\pi(x)})^\perp$  are  $g_t$ -orthogonal planes with zero  $g_t$ -curvature at  $(p_1, \pm p_1) \in \pm \Delta S^2$ , thus  $\sec_{g_t}^\perp(\sigma_{\pi(x)}) = 0$ . Moreover, a plane  $\sigma$  has  $\sec_{g_t}^\perp(\sigma) = 0$  if and only if it is of this form.*

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<sup>1</sup>i.e., this plane is spanned by one vertical and one horizontal vector.

*Proof.* The fact that  $\sigma_{\pi(x)}$  and  $\sigma_{\pi(x^\perp)} = (\sigma_{\pi(x)})^\perp$  are  $g_t$ -orthogonal planes for  $t > 0$  follows immediately from (2.4) and (2.9). Furthermore, they have zero  $g_t$ -curvature as a consequence of Proposition 2.1. All other claims follow from the above discussion and Corollary 2.2.  $\square$

*Remark 2.4.* For  $n \geq 3$ , although there exists an analogous cohomogeneity one  $\mathrm{SO}(n+1)$ -action on  $S^n \times S^n$ , the corresponding Cheeger deformation fails to produce so many positively curved planes. This is due to the fact that  $\mathrm{SO}(n+1)$ ,  $n \geq 3$ , is not positively curved, see Corollary 2.2. As a result, even if we consider a more general definition of biorthogonal curvature (as a function of two planes) for higher dimensions, this step in our construction would still only work on  $S^n \times S^n$  if  $n = 2$ .

### 3. SECOND STEP: FIRST-ORDER LOCAL CONFORMAL DEFORMATION

As seen above, the Cheeger deformed metrics  $g_t$ ,  $t > 0$ , have  $\sec_{g_t}^\perp \geq 0$  and equality holds only for certain planes (of the form (2.8)) at  $\pm\Delta S^2$ . In order to get these planes to also have  $\sec^\perp > 0$ , we now carry out a (local) first-order conformal deformation, inspired by results of Strake [9]. More precisely, choose  $g$  to be a Cheeger deformed metric  $g_t$  for any  $t > 0$  and consider the new 1-parameter family

$$(3.1) \quad g_s := g + s h, \quad s \in ]-\varepsilon, \varepsilon[,$$

where  $h$  is some symmetric  $(0, 2)$ -tensor to be defined, and  $\varepsilon > 0$  is small enough so that  $g_s$  is still a Riemannian metric. Given the above geometry of the Cheeger deformed metric, we will choose  $h$  such that  $\frac{d}{ds} \sec_{g_s}^\perp(\sigma)|_{s=0} > 0$  for all planes  $\sigma$  with zero  $g$ -curvature. The crucial observation that makes this possible is that these planes are always transverse to  $\pm\Delta S^2$ . Our choice will be such that  $h$  is supported only near  $\pm\Delta S^2$  and is pointwise proportional to  $g$ , justifying the terminology. Finally, since the set of zero  $g$ -curvature planes is compact in  $Gr_2 TM$  and  $\sec_g^\perp \geq 0$ , a standard argument then guarantees that  $\sec_{g_s}^\perp$  for every  $s > 0$  sufficiently small

We start by recalling the first variation of  $\sec_{g_s}(\sigma)$ , see Strake [9, Sec 3.a].

**Proposition 3.1.** *Let  $(M, g)$  be a Riemannian manifold with  $\sec_g \geq 0$  and  $X, Y \in T_p M$  be  $g$ -orthonormal vectors that span a  $g$ -flat plane  $\sigma \subset T_p M$ . Consider a first-order variation  $g_s = g + s h$ . Then the first variation of  $\sec_{g_s}(\sigma)$  is given by*

$$\left. \frac{d}{ds} \sec_{g_s}(\sigma) \right|_{s=0} = \nabla_X \nabla_Y h(X, Y) - \frac{1}{2} \nabla_X \nabla_X h(Y, Y) - \frac{1}{2} \nabla_Y \nabla_Y h(X, X).$$

In particular, if  $h = \phi g$ , then

$$(3.2) \quad \left. \frac{d}{ds} \sec_{g_s}(\sigma) \right|_{s=0} = -\frac{1}{2} \mathrm{Hess} \phi(X, X) - \frac{1}{2} \mathrm{Hess} \phi(Y, Y).$$

Now, observe that if  $N \subset M$  is an embedded submanifold, the squared distance function  $\psi(p) = \mathrm{dist}(p, N)^2$  is smooth in a sufficiently small tubular neighborhood of  $N$ . The gradient of  $\psi$  at  $p$  vanishes if  $p \in N$ , and points in the outward radial direction if  $p \notin N$ . The Hessian of  $\psi$  at  $p \in N$  is given by:

$$(3.3) \quad \mathrm{Hess} \psi(X, X) = 2g(X_\perp, X_\perp) = 2\|X_\perp\|_g^2, \quad X \in T_p M,$$

where  $X = X_\top + X_\perp \in T_p N \oplus (T_p N)^\perp$  is the  $g$ -orthogonal decomposition in tangent and normal parts to  $N$ . We are now ready for the last step in the deformation process.

**Proposition 3.2.** *Consider the metrics  $g_s$  on  $S^2 \times S^2$ , given by (3.1). There exists a smooth function  $\phi: S^2 \times S^2 \rightarrow \mathbb{R}$ , supported in a neighborhood of  $\pm\Delta S^2$ , such that if  $h = \phi g$ , then  $\frac{d}{ds} \sec_{g_s}^\perp(\sigma)|_{s=0} > 0$  for all planes  $\sigma$  such that  $\sec_g^\perp(\sigma) = 0$ . In particular,  $\sec_{g_s}^\perp > 0$  for any sufficiently small  $s > 0$ .*

*Proof.* From Proposition 2.3, the only planes  $\sigma$  with  $\sec_g^\perp(\sigma) = 0$  are of the form  $\sigma_{\pi(x)} \subset T_{(p_1, \pm p_1)}(S^2 \times S^2)$  for some  $x \in \{p_1\}^\perp$ . These planes are vertical, i.e., they contain a direction normal to  $\pm\Delta S^2$ .

Let us analyze the plane  $\sigma_{\pi(x)}$  at  $(p_1, p_1) \in \Delta S^2$ , the case of  $(p_1, -p_1) \in -\Delta S^2$  being totally analogous. As mentioned above, the function  $\psi_+(p) = \text{dist}(p, \Delta S^2)^2$  is smooth in a tubular neighborhood  $D(\Delta S^2)$ . Let  $\chi_+: S^2 \times S^2 \rightarrow \mathbb{R}$  be a smooth cutoff function that vanishes outside  $D(\Delta S^2)$  and is equal to 1 in a smaller tubular neighborhood of  $\Delta S^2$ . Then, according to (2.8), (3.2) and (3.3), if we set  $h_+ = -(\chi_+ \psi_+) g$  and  $g_s^+ = g + s h_+$ , then

$$\frac{d}{ds} \sec_{g_s^+}^\perp(\sigma_{\pi(x)}) \Big|_{s=0} = \|(X_{p_1}^*, -X_{p_1}^*)\|_g^2 > 0.$$

Clearly, the same holds for  $\sigma_{\pi(x)}^\perp$ . In particular,

$$\frac{d}{ds} \sec_{g_s}^\perp(\sigma_{\pi(x)}) \Big|_{s=0} = \frac{1}{2} \left( \frac{d}{ds} \sec_{g_s^+}^\perp(\sigma_{\pi(x)}) \Big|_{s=0} + \frac{d}{ds} \sec_{g_s^+}^\perp(\sigma_{\pi(x)}^\perp) \Big|_{s=0} \right) > 0.$$

Defining  $\psi_-$  and  $\chi_-$  analogously, we get  $h_- = -(\chi_- \psi_-) g$  and  $g_s^- = g + s h_-$  with the same property as above for planes at  $(p_1, -p_1) \in -\Delta S^2$  with zero  $g$ -biorthogonal curvature. Thus, the function  $\phi := -(\chi_+ \psi_+) - (\chi_- \psi_-)$  has the desired properties. More precisely,  $h = \phi g = h_+ + h_-$  is such that  $g_s = g + s h$  coincides with  $g_s^\pm$  near  $\pm\Delta S^2$ , hence  $\frac{d}{ds} \sec_{g_s}^\perp(\sigma)|_{s=0} > 0$  for all planes  $\sigma$  such that  $\sec_g^\perp(\sigma) = 0$ .

Finally, we claim that this implies  $g_s$  has positive biorthogonal curvature for  $s > 0$  sufficiently small. Let  $U \supset \pm\Delta S^2$  be a relatively compact neighborhood of  $\pm\Delta S^2$  in  $S^2 \times S^2$ , such that  $g_s = g$  on  $S^2 \times S^2 \setminus \overline{U}$ . Let

$$\mathcal{D}_\varepsilon := \{\sigma \in Gr_2 T\overline{U} : \sec_g^\perp(\sigma) \leq \varepsilon\}, \quad \mathcal{D}^\varepsilon := \{\sigma \in Gr_2 T\overline{U} : \sec_g^\perp(\sigma) \geq \varepsilon\}.$$

Notice that for every  $\varepsilon \geq 0$ , the sets  $\mathcal{D}_\varepsilon$  and  $\mathcal{D}^\varepsilon$  are compact. For simplicity, write

$$f: [0, S] \times Gr_2 T\overline{U} \rightarrow \mathbb{R}, \quad f(s, \sigma) = \sec_{g_s}^\perp(\sigma).$$

This function is smooth and, by the above,  $\frac{\partial f}{\partial s} \Big|_{\{0\} \times \mathcal{D}_0} > 0$ . By compactness of  $\mathcal{D}_0$ , we get that  $\frac{\partial f}{\partial s} \Big|_{\{0\} \times \mathcal{D}_0} > \delta > 0$ . By continuity of  $\frac{\partial f}{\partial s}$ , there exists  $\varepsilon > 0$  such that  $\frac{\partial f}{\partial s} \Big|_{\{0\} \times \mathcal{D}_\varepsilon} > 0$ . Thus, if  $\sigma \in \mathcal{D}_\varepsilon$ , there exists  $s_\sigma > 0$  such that  $f(s, \sigma) > 0$  for all  $s \in ]0, s_\sigma]$ . Let  $s_* := \inf\{s_\sigma : \sigma \in \mathcal{D}_\varepsilon\}$ , which is positive by compactness of  $\mathcal{D}_\varepsilon$ . If  $\sigma \in \mathcal{D}^\varepsilon$ , by continuity of  $f$ , there exists  $s^\sigma > 0$  such that  $f(s, \sigma) > 0$  for all  $s \in ]0, s^\sigma]$ . Let  $s^* := \inf\{s^\sigma : \sigma \in \mathcal{D}^\varepsilon\}$ , which is positive by compactness of  $\mathcal{D}^\varepsilon$ . Set  $s_1 := \min\{s_*, s^*\} > 0$ . It is now clear that  $f(s, \sigma) = \sec_{g_s}^\perp(\sigma) > 0$  for all  $\sigma \in Gr_2 T\overline{U}$  and  $s \in ]0, s_1]$ . Since  $g = g_s$  outside  $\overline{U}$ , it follows that  $\sec_{g_s}^\perp > 0$  globally.  $\square$

This concludes the proof of the Theorem in the Introduction. Moreover, it follows immediately from the above results (Propositions 2.3 and 3.1) that metrics with positive biorthogonal curvature on  $S^2 \times S^2$  can be constructed arbitrarily close to the standard product metric  $g_0$ , in any  $C^k$ -topology,  $k \geq 1$ . In particular, these metrics also have positive Ricci curvature.



*Remark 3.3.* The above first-order deformation  $g_s$  works to get  $\sec^\perp > 0$  on all of  $S^2 \times S^2$  because the only points that have planes where  $\sec_g^\perp = 0$  are contained in the submanifolds  $\pm \Delta S^2$ , which admit a relatively compact neighborhood and where we have  $\frac{d}{ds} \sec_{g_s}^\perp(\sigma)|_{s=0} > 0$  for all  $\sigma$  with  $\sec_g^\perp(\sigma) = 0$ . The same cannot be done for the sectional curvature because at *every point* there is a plane  $\sigma$  with  $\sec_g(\sigma) = 0$ . Thus, the only type of first-order deformation that would give  $\sec_{g_s} > 0$  would be one with  $\frac{d}{ds} \sec_{g_s}(\sigma)|_{s=0} > 0$  for all  $\sigma$  with  $\sec(\sigma) = 0$ . It was proved by Strake [9, Prop. 4.3] that such a deformation of first-order does not exist on  $(S^2 \times S^2, g)$  due to the presence of totally geodesic flat tori.

#### 4. FINAL REMARKS

**4.1. Biorthogonal pinching and isotropic curvature.** The biorthogonal curvature of a manifold  $(M, g)$  is said to be *(weakly)  $1/4$ -pinched* if there exists a positive function  $\delta$  such that  $\frac{\delta}{4} \leq \sec_g^\perp(\sigma) \leq \delta$  for all  $\sigma$ . This notion can be extended to any dimensions by requiring that the average of any two mutually orthogonal planes is  $1/4$ -pinched. As observed by Seaman [8], this pinching condition implies that the manifold has nonnegative isotropic curvature. It was later proved by Seaman [7], and independently by Micallef and Wang [3], that if an even dimensional compact orientable manifold  $(M, g)$  with  $b_2(M) \neq 0$  has nonnegative isotropic curvature and positive biorthogonal curvature at one point, then  $(M, g)$  is Kähler,  $b_2(M) = 1$  and  $M$  is simply-connected. Consequently, our metrics  $g_*$  of positive biorthogonal curvature on  $S^2 \times S^2$  cannot satisfy the biorthogonal  $1/4$ -pinching condition, since  $b_2(S^2 \times S^2) = 2$ . Moreover, it also follows that  $g_*$  does not have nonnegative isotropic curvature.

**4.2. Negative sectional curvatures.** Although the first step in our deformation preserves  $\sec \geq 0$  from the product metric, the second step does not. In fact, even though  $\sec_{g_*}^\perp > 0$ , there are planes  $\sigma$  in  $(S^2 \times S^2, g_*)$  with  $\sec_{g_*}(\sigma) < 0$ . This follows from an obstruction to positive first-order deformations observed by Strake [9, Sec. 4]. Namely, all zero planes in the Cheeger deformed metric  $g = g_t$  from Section 2 are tangent to a totally geodesic flat torus, see Mütter [4, Satz 4.26]. Pick one such torus  $i: T^2 \hookrightarrow (S^2 \times S^2, g)$ , that intersects  $\pm \Delta S^2$ . The first-order deformation  $g_s = g + s h$  on  $S^2 \times S^2$  induces a first-order deformation  $i^* g_s$  on  $T^2$ . As observed by Strake [9, Lemma 4.1], since  $i(T^2)$  is totally geodesic, the first variation for the sectional curvature on  $T^2$  coincides with the ambient variation:

$$(4.1) \quad \frac{d}{ds} \sec_{i^* g_s}(\sigma) \Big|_{s=0} = \frac{d}{ds} \sec_{g_s}(di(\sigma)) \Big|_{s=0}.$$

In fact, this follows directly by differentiating the Gauss equation of  $i(T^2) \subset (S^2 \times S^2, g_s)$  at  $s = 0$ . Let  $i(p)$  be a point where  $i(T^2)$  intersects  $\pm \Delta S^2$ . Then if  $\sigma = T_p T^2$ ,  $di(\sigma)$  is such that  $\sec_g^\perp(di(\sigma)) = 0$ , so the construction in Proposition 3.1 is such that (4.1) is positive. By the Gauss-Bonnet Theorem,  $A(s) = \int_{T^2} \sec_{i^* g_s} \text{vol}_{i^* g_s} = 2\pi\chi(T^2)$  vanishes identically, so that

$$(4.2) \quad 0 = A'(0) = \int_{T^2} \left( \frac{d}{ds} \sec_{i^* g_s} \Big|_{s=0} \right) \text{vol}_{i^* g}.$$

Since the above integrand is positive at  $i(p) \in \pm \Delta S^2$ , it must also be negative somewhere. Together with (4.1) and the fact that  $i(T^2) \subset (S^2 \times S^2, g)$  is totally geodesic and flat, this means that  $g_s$  must have some negative sectional curvature.



**4.3. Modified Yamabe invariant.** As observed by Costa [2], the minimum of the biorthogonal curvature at each point is a *modified scalar curvature*, with corresponding *modified Yamabe invariant* denoted by  $Y_1^\perp(M) = \sup_g Y_1^\perp(M, g)$ , where the supremum is taken over all metrics  $g$  on  $M$ . It is observed that if a metric  $g \in [g_0]$  is conformal to the standard product metric on  $S^2 \times S^2$ , then  $Y_1^\perp(S^2 \times S^2, g) \leq 0$ . In particular, no metric conformal to  $g_0$  can have positive biorthogonal curvature. However, as a direct consequence of the Theorem in the Introduction, we have that  $Y_1^\perp(S^2 \times S^2) > 0$ , see [2, Thm 3 (1)].

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